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PRIMARY PRESSURE WAVE IN A FLUID AFTER ACTUATION OF A PIPELINE VALVE

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This paper considers the variation in the velocity, density, and pressure of an inviscid compressible fluid due to flow acceleration or deceleration after a change in the flow area of a valve installed on the pipeline with rigid walls. Expressions are given for the amplitude of the primary compression or rarefaction wave resulting form the change in flow area of the valve.

Key words: *inviscid compressible fluid, primary pressure wave, pipeline, valve.*

The primary pressure wave due to the occurrence or cessation of a fluid flow (hydraulic impact) has been the subject of extensive studies, in particular [1–3]. Unlike in these papers, in the present study, we use the analytic solutions of the corresponding nonlinear partial differential equations obtained in our studies for various boundary conditions.

1. Formulation of the Problems. Actuation of a valve installed in a pipeline filled with a quiescent or moving compressible fluid gives rise to a primary wave of reduced or elevated pressure. Let us consider unsteady isentropic flow of an inviscid compressible fluid in an inclined pipeline of constant cross section with rigid impermeable walls for the case of noninstantaneous actuation of the valve. It is assumed that the fluid moves at a velocity u that depends only on the longitudinal coordinate z and time t. In this case, the continuity and dynamic equations for continuous media [4] become

$$R\,\frac{\partial\rho}{\partial t}+\frac{\partial\,(\rho u)}{\partial z}=0; \tag{1.1}$$

$$R\rho\,\frac{\partial u}{\partial t}+\rho u\,\frac{\partial u}{\partial z}+\frac{\partial P}{\partial z}=R\rho F_z; \tag{1.2}$$

$$\frac{\partial P}{\partial r}=R\rho F_r; \tag{1.3}$$

$$\frac{1}{r}\,\frac{\partial P}{\partial\varphi}=R\rho F_\varphi. \tag{1.4}$$

Here ρ and P are the fluid density and pressure, R is the pipeline radius, r and z are the radial and longitudinal coordinates divided by R, φ is the angular coordinate, ρF_z, ρF_r, and ρF_φ are the projections of the body force gradient onto the z, r, and φ axes, respectively.

If the body forces are ignored or a vertical pipeline is considered, the right sides of Eqs. (1.3) and (1.4) vanish and the velocity, density, and pressure of the inviscid compressible fluid do not depend on r and φ. In this system (1.1), (1.2) should be supplemented by the equation of state $P=f(\rho)$ or the equation $\partial P/\partial z=A^2\,\partial\rho/\partial z$ for $A=\text{const}$.

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Let us show that in the case of action of only gravity, i.e.,

$$\rho F_z = \rho g \sin\theta, \qquad \rho F_r = \rho g \cos\theta \cos\varphi, \qquad \rho F_\varphi = -\rho g \cos\theta \sin\varphi, \tag{1.5}$$

where θ is the pipeline slope to the horizon and g is the acceleration of gravity; the required quantities ρ, P, and u are uniquely determined by system (1.1)–(1.4), despite its seeming overdetermination.

The quantity φ is reckoned clockwise from the direction to the vertex of the pipeline flow area, and the coordinates z is reckoned from the valve in the direction of propagation of the pressure wave.

Below, we use the dimensionless quantities

$$\tau = \frac{at}{R}, \qquad M = \frac{u}{a}, \qquad z_* = \frac{L}{R}, \qquad B = \frac{Rg\sin\theta}{a^2}, \tag{1.6}$$

where L is the pipeline length and a is the sound velocity in the fluid for $\tau = 0$, $z = 0$, and $r = 0$.

1.1. Along with Eqs. (1.1)–(1.4) and equalities (1.5), the problem of acceleration of the inviscid compressible flow after the beginning of valve opening includes:

— the initial conditions

$$\rho(0, z) = \rho_0(z), \qquad M(0, z) = M_0(z) \tag{1.7}$$

[in the particular case, $M_0(z) = 0$];

— the boundary condition

$$z = 0: \qquad M(\tau, 0) = f_1(\tau), \quad f_1(\tau_k) \equiv M_{k,0}, \quad f_1(0) \equiv f_{1,0}, \tag{1.8}$$

where $\tau = 0$ and τ_k correspond to the moments of the beginning and termination of valve opening;

— the conditions for the coordinate of the beginning of the primary pressure wave (PPW) front $z_p(\tau)$ at $z_p \leqslant z_*$, which coincide with the initial conditions for the corresponding cross sections:

$$M(\tau, z_p) = M_0(z_p), \qquad \rho(\tau, z_p) = \rho_0(z_p). \tag{1.9}$$

The coordinate $z_p(\tau)$ for $\tau \leqslant \tau_2$ is defined by the equation

$$\tau = \int_0^{z_p} \frac{dz}{1 + M_0(z)}, \tag{1.10}$$

and the travel time of the beginning of the PPW in the pipeline equals

$$\tau_2 = \int_0^{z_*} \frac{dz}{1 + M_0(z)}. \tag{1.11}$$

In the particular case,

$$z_p = \begin{cases} \tau, & M_0(z_p) = 0; \\ z_* = \text{const}, & \tau > \tau_2. \end{cases} \tag{1.12}$$

1.2. Along with Eqs. (1.1)–(1.5), the problem of deceleration the inviscid compressible flow after the beginning of valve closure includes:

— the initial conditions (1.7) at $M_0(z) \neq 0$;

— the boundary conditions

$$z = 0: \qquad M(\tau, 0) = f_2(\tau), \quad f_2(0) \equiv M_{0,0}, \quad f_2(\tau_k) \equiv M_{k,0}. \tag{1.13}$$

Here $M_{0,0} \neq 0$ and τ_k corresponds to the moment of the completion of valve flow area reduction from the initial to the final value; in the particular case (complete closure of the valve), $M_{k,0} = 0$;

— the conditions for the coordinates of the beginning of the PPW front z_p, which coincide with the initial conditions for the corresponding cross sections and, formally, with conditions (1.9)–(1.12).

The nature of the functions $f_1(\tau)$ and $f_2(\tau)$ in conditions (1.8) and (1.13) depends on the downstream pressure and the features of the valve (its design and response of the operating mechanism).

We find analytic solutions of Eqs. (1.1)–(1.5) subject to conditions (1.7)–(1.12) for $\tau \leqslant z_*$ and with conditions (1.9)–(1.13) and (1.7) for $M_0(z) \neq 0$, $0 \leqslant \tau < z_*/(1 + M_{0,0})$, and $0 < |M| \leqslant 1$ for one kind of the functions $f_1(\tau)$ and $f_2(\tau)$ that contain several arbitrary constants.

2. Analytic Solutions. Differentiating (1.3) with respect to φ and (1.4) with respect to r, eliminating P, and taking into account (1.5), we obtain the equations

$$Rg\cos\theta\Big(r\,\frac{\partial\rho}{\partial r}\,\sin\varphi+\frac{\partial\rho}{\partial\varphi}\,\cos\varphi\Big)=0.$$

From this we have

$$\rho=\rho_{0,0}f_3(\tau,z)F(x),\qquad x\equiv(Rg/a^2)r\cos\theta\cos\varphi, \tag{2.1}$$

where $F(x)$ is an arbitrary function x; $\rho_{0,0}$ is the density of the inviscid compressible fluid for $\tau=0$, $z=0$, and $r=0$. Along with (2.1), system (1.3)–(1.5) corresponds to the expression

$$P=\rho_{0,0}a^2\Big[f_4(\tau,z)+f_3(\tau,z)\int F(x)\,dx\Big], \tag{2.2}$$

where f_3 and f_4 are arbitrary functions of τ and z.

Expression (2.1), which completely corresponds to system (1.3)–(1.5), does not contradict Eq. (1.1) only for $u\neq f(r,\varphi)$. According to (1.2) with (1.5), this implies the condition $\rho^{-1}\,\partial P/\partial z\neq f(r,\varphi)$. Therefore, expressions (2.1) and (2.2) correspond to the entire system (1.1)–(1.5) only for $\partial f_4/\partial z=0$ and $\dfrac{1}{F(x)}\displaystyle\int F(x)\,dx\neq f(x)$. Use $F(x)=\exp(x/\beta^2)$ for $\beta=\mathrm{const}$. Then, system (1.1)–(1.6) reduces to the following system:

$$\begin{gathered}\frac{\partial\Phi}{\partial\tau}+\frac{\partial M}{\partial z}+M\,\frac{\partial\Phi}{\partial z}=0,\qquad \Phi=\ln f_3(\tau,z),\\ \frac{\partial M}{\partial\tau}+M\,\frac{\partial M}{\partial z}=B-\beta^2\,\frac{\partial\Phi}{\partial z},\qquad M\neq f(r,\varphi);\end{gathered} \tag{2.3}$$

$$P=a^2[f_4(\tau)\rho_{0,0}+\beta^2\rho],\qquad \beta=\mathrm{const}; \tag{2.4}$$

$$\rho=\rho_{0,0}f_3(\tau,z)\exp(x/\beta^2). \tag{2.5}$$

We note that expressions (2.4) and (2.5) imply that $\partial P/\partial z=a^2\beta^2\,\partial\rho/\partial z$.

The conditions of the formulated problems satisfy the particular solution of system (2.3)

$$f_3(\tau,z)=C\exp[(M-B\tau)/b],\qquad C=\mathrm{const}; \tag{2.6}$$

$$\frac{\partial M}{\partial\tau}+(M+b)\,\frac{\partial M}{\partial z}=B,\qquad b=\pm\beta, \tag{2.7}$$

in which, according to (1.6), $0\leqslant B\ll 1$ (for a horizontal pipeline, $B=0$), and Eq. (2.7) is satisfied by the following two expressions for M:

$$M(\tau,z)=[k_2+(k_3+z)y+c_0\sqrt{h}\,]/T-b+By/2 \tag{2.8}$$

$[h=k_4(k_3+z)^2+2k_2(k_3+z)y+k_5T+k_2^2+k_4B^2T^2/4-BT(k_4z+k_2y),\ T=y^2-k_4,\ y=k_1+\tau,\ k_i=\mathrm{const},\ c_0=\pm1]$

and

$$M(\tau,z)=c_2+(c_3+z)/y-b+3By/8+c_0\sqrt{H} \tag{2.9}$$

$$[H=[(c_3+z)/y-c_2-By/8]^2+c_4/y,\quad y=c_1+2\tau,\quad c_i=\mathrm{const},\quad c_0=\pm1].$$

Substitution of expressions (2.8) and (2.9) brings Eq. (2.7) to identity. According to (2.5) and (2.6), after finding the constant C, we obtain

$$\rho=\rho_{0,0}\exp[x/b^2+(M-M_{0,0}-B\tau)/b]. \tag{2.10}$$

These solutions supplement the solutions known for the problems of unsteady fluid flow in a pipeline [5, 6].

3. Primary Pressure Wave Due to Acceleration of Inviscid Compressible Flow after Valve Opening. In this case, if the directions of the PPW and the inviscid compressible flow coincide, $M>0$ and $b>0$, which corresponds to the initial stage of pipeline filling. If the directions of the PPW and the inviscid compressible flows are opposite, then $M<0$ and $b>0$ (beginning of pipeline evacuation).

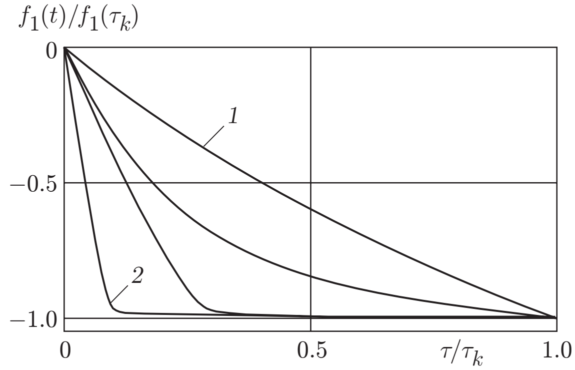

Fig. 1. Possible velocity distributions at the valve during its opening.

Expressions (2.4), (2.8), and (2.10) are the solution of the problem (1.1)–(1.5) subject to the conditions (1.7)–(1.10), $z < z_*$, $\tau < \tau_*$, and

$$f_1(\tau) = By/2 - b + (k_2 + k_3 y + c_0\sqrt{h_0}\,)/T, \tag{3.1}$$

where $h_0 = 2k_2k_3y + k_3^2k_4 + k_5T + k_2^2 + BT(Bk_4T/4 - k_2y)$, and if the initial conditions $\rho_0(z)$ and $M_0(z)$ in (1.7) satisfy the relation

$$\rho_0(z) = \rho_{0,0}\exp\,[(M_0(z) - M_{0,0})/b]. \tag{3.2}$$

Some of the constants in expressions (2.8) and (3.1) are found from condition (1.9) taking into account (1.10). For example, for $M_0(z) = 0$, according to (3.2), we have $\rho_0(z) = \rho_{0,0}$, which, from physical considerations, is possible under gravity only for $B = 0$. In this case,

$$b = 1, \qquad k_5 = (k_3 - k_1)^2 - k_4 - 2k_2. \tag{3.3}$$

These relations also satisfy condition (1.7) and are obtained after the substitution $z = z_p$, $z_p = \tau$, and $M = 0$ in (2.8).

For $\tau \leqslant \tau_k$, the coordinate of the end of the PPW front z_1 is zero, and for $\tau_k < \tau \leqslant z_*$,

$$z_1 = \int\limits_{\tau_k}^{\tau} (1 + M)\,d\tau \approx (1 + M_{k,0})(\tau - \tau_k). \tag{3.4}$$

According to (2.4) and (2.10), the PPW amplitude in an inviscid compressible fluid is equal to

$$AD = E\exp\Big(\frac{x}{b^2} - \frac{B\tau}{b}\Big)\Big[\exp\Big(\frac{M(\tau, z_1)}{b}\Big) - \exp\Big(\frac{M_0(z_p)}{b}\Big)\Big],$$

where $AD = P(\tau, z_1) - P(\tau, z_p)$ and $E = \rho_{0,0}a^2b^2$. The amplitude depends on z, τ_k, $M_{k,0}$, and on the constant included in expressions (2.8) and (3.1).

It is evident that expression (3.1) contains several integration constants. By varying them, it is possible to find a solution of the problem for a wide range of boundary conditions (also nonarbitrary) $f_1(\tau)$ in (1.8).

For example, it is possible to uniquely relate the constants $k_1, \ldots, k_4$ to the values of $M(\tau_k, 0)$, $M(\tau_k/2, 0)$ and the derivative M with respect to τ at the points $z = 0$, $\tau = 0$ and $z = 0$, $\tau = \tau_k$, i.e., to the properties and actuation regime of the valve and the downstream pressure. Figure 1 gives plots of the functions $f_1(\tau)/|f_1(\tau_k)|$ calculated for $b = 1$, $c_0 = 1$, $B = 0$, $M_0(z) = 0$, and a zero derivative of M with respect to τ for $z = 0$ and $\tau = \tau_k$ [i.e., under the assumption of a smooth variation of the function $f_1(\tau)$ in the stage of completion of valve opening and the condition $f_1(\tau > \tau_k) = \text{const}$ for $\tau < z_*$]. In particular, curve 1 is plotted for $k_1 = 3.95$, $k_2 = 3.96$, $k_3 = 0.178$, $k_4 = -0.198$, and $k_5 = -0.159$, and curve 2 is plotted for $k_1 = 213.05$, $k_2 = -3.626\cdot 10^4$, $k_3 = 170.86$, $k_4 = 4.523\cdot 10^4$, and $k_5 = 2.91\cdot 10^4$.

For $M < 0$ and conditions (1.7), the initial segment of the PPW front propagates against the flow moving at velocity $M_0(z)$ and the final segment propagates against the flow with a larger modulus of the velocity $M_{k,0}$.

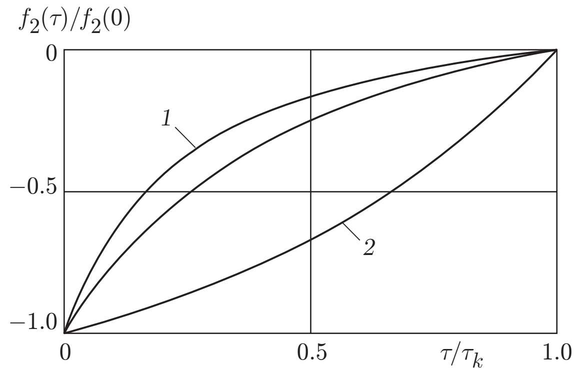

Fig. 2. Possible velocity distributions at the valve during its closure.

Therefore the time of travel of the final segment to a certain cross section z is larger than that for the initial segment of the PPW front.

Generally, for $\tau_k \leqslant \tau \leqslant z_*$, the length of the PPW front is defined as

$$z_p - z_1 \approx \tau(1 + M_{0,0}) - (\tau - \tau_k)(1 + M_{k,0}), \tag{3.5}$$

and for $\tau_k > z_*/(1 + M_{0,0})$, it equals z_*.

According to (3.5), for $\tau \leqslant z_*$, the PPW propagation in the pipeline due to valve opening is accompanied by extension of the PPW front for $M < 0$ and by compression for $M > 0$.

4. Primary Pressure Wave for Flow Deceleration due to Valve Closure. In the case where the PPW propagates over an inviscid compressible flow, $M > 0$ and $b > 0$ (the beginning of termination of pipeline filling). If the propagation directions of the PPW and the flows are opposite, then $M < 0$ and $b > 0$ (a hydraulic impact due to termination of the inviscid compressible flow from the pipeline takes place).

Expressions (2.4), (2.9), and (2.10) are the solution of the problem (1.1)–(1.6) subject to conditions (1.7), (1.9)–(1.13), (3.2), $z \leqslant z_*$, $\tau \leqslant z_*/(1 + M_{0,0})$, and

$$f_2(\tau) = c_2 + c_3/y + 3By/8 - b + c_0\sqrt{(c_3/y - c_2 - By/8)^2 + c_4/y}.$$

The constant c_4 is found from condition (1.12) for $\tau = \tau_k$:

$$c_4 = (M_{k,0} + b)[(M_{k,0} + b - 2c_2)y_k - 3By_k^2/4 - 2c_3] + (4c_2 + By_k)(c_3 + By_k^2/8),$$

where $y_k = 2\tau_k + c_1$.

The constants c_1 and c_3 are determined from condition (1.9) with allowance for (3.3). For example, for $\rho_0(z) = \rho_0(0)$ and $M_0(z) = M_{0,0}$ (which is possible only in a horizontal pipeline), we have $B = 0$ and, according to (1.10), $z_p = \tau(1+M_{0,0})$. Substituting $z = z_p$ and $M = M_{0,0}$ into equality (2.9), we obtain the equation for τ, which becomes identity for various τ if the sums of terms proportional to τ and terms independent of τ are set equal to zero, i.e., if the following conditions are satisfied:

$$(1 - b)(2c_2 - b - M_{0,0}) = 0, \qquad c_4 = (2c_3 - c_1 - c_1M_{0,0})(2c_2 - b - M_{0,0}).$$

Figure 2 gives plots of the functions $f_2(\tau)/|f_2(0)|$ calculated for $b = 1$ and $c_0 = 1$ and constants $c_1 = 0.562$, $c_2 = 0.577$, $c_3 = -0.188$, and $c_4 = -0.236$ (curve 1) and $c_1 = -13.103$, $c_2 = -5.562$, $c_3 = -61.841$, and $c_4 = 676.962$ (curve 2) for which expressions (2.4) and (3.1) are the solution of the problem (1.1)–(1.5), (1.7), (1.9)–(1.13).

According to (2.4), (2.9), and (2.10) for each value of $z \leqslant z_*$, the amplitude of the primary pressure wave is given by

$$AD = E\exp\Big(\frac{f_2(\tau) - M_{0,0}}{b} + \frac{x}{b^2} - \frac{B}{b}\,\tau\Big)[1 - \exp{(DM)}], \tag{4.1}$$

where $AD = P(\tau, 0) - P(\tau, z)$, $DM = (M - f_2(\tau))/b$, and $E = \rho_{0,0}a^2b^2$.

For $B = 0$,

$$DM = \frac{1}{b}\Big[\frac{z}{y} \pm \sqrt{\Big(\frac{c_3 + z}{y} - c_2\Big)^2 + \frac{c_4}{y}} \mp \sqrt{\Big(\frac{c_3}{y} - c_2\Big)^2 + \frac{c_4}{y}}\Big]. \qquad (4.2)$$

In the case $M < 0$ and $b > 0$ (hydraulic impact), the coordinate of the beginning of the PPW front z_p is determined according to (1.10) and (1.11). In particular, if $M_0(z) = M_{0,0}$, then

$$z_p = \tau(1 + M_{0,0}), \qquad \tau_2 = z_*/(1 + M_{0,0}).$$

For $\tau \leqslant \tau_k$, the coordinate of the end of the PPW front z_1 is equal to zero, and for $\tau_k < \tau \leqslant \tau_2$ it is determined according to (3.4).

Therefore, for $\tau_2 \leqslant \tau_k$ the length of the PPW front is $z_p - z_2 = z_*$. For $f_2(\tau_k) = 0$ (complete valve closure) and $\tau_k < \tau_2$, the length of the PPW front is $z_p - z_2 \approx \tau M_{0,0} + \tau_k$.

The amplitude of the primary pressure wave is maximum for $\tau_k \leqslant \tau_2$ and $f_2(\tau_k) = 0$, and at the time $\tau = \tau_k$, it reaches the value

$$AD_{\max} = E \exp\Big(\frac{x}{b^2} - \frac{B}{b}\,\tau_k\Big)\Big[\exp\Big(-\frac{M_{0,0}}{b}\Big) - 1\Big]. \qquad (4.3)$$

For $\tau_k > \tau_2$, AD is maximum at the time $\tau = \tau_2$, reaching the value

$$AD_{\max} = E \exp\Big(\frac{x}{b^2} - \frac{B}{b}\,\tau_2\Big)\Big[\exp\Big(\frac{f_2(\tau_2) - M_{0,0}}{b}\Big) - 1\Big]. \qquad (4.4)$$

From (4.3) it follows that as a first approximation for $B = 0$, $|M_{0,0}| \ll 1$, $x = 0$ and $M_{0,0} < 0$ (the PPW for valve closure), we have

$$AD_{\max} = \exp\Big(-\frac{M_{0,0}}{b}\Big)\rho_{0,0} a |u_{0,0}| b\Big[1 + \frac{1}{2b}\,M_{0,0} + \frac{1}{6b^2}\,(M_{0,0})^2 + \ldots\Big]. \qquad (4.5)$$

This value is close to the result $AD_{\max} = \rho_{0,0} a |u_{0,0}|$ obtained by Joukowski [1] for $b = 1$ and the conditions of validity of formula (4.5) using a different method [taking into account the compliance of the pipeline walls and assuming a linear decrease in the velocity $u(0,t)$ or instantaneous blocking of the pipeline] without using Eqs. (1.1), (1.3), and (1.4).

An analysis of expressions (4.1)–(4.4) shows that at the moment of closure of a gate or a valve installed in a pipeline with rigid walls, the PPW amplitude:

— is proportional to $\rho_{0,0} a$;

— increases with increase in the difference between the final (at the moment of completion of valve operation τ_k) and initial velocities at the valve;

— has a maximum at the time when $M_{k,0} = 0$;

— does not depend on the operation time τ_k and z if $\tau_k \leqslant \tau_2$ and $\theta = 0$;

— decreases considerably with increase in $f_2(\tau_2)$ if $\tau_k > \tau_2$, i.e., depends on the duration and nature of valve closure and downstream pressure.

Since these conclusions are consistent with the well-known experimental data [1], the assumptions used in the formulation of the problem are of little significance.

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